ELSEVIER Journal of Geometry and Physics 29 (1999) 5–12

JOURNAL OF GEOMETRY AND PHYSICS

The elliptic genus of Calabi–Yau 3- and 4-folds, product formulae and generalized Kac–Moody algebras

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Received 13 March 1997

Abstract

In this paper the elliptic genus for a general Calabi–Yau 4-fold is derived. The recent work of Kawai calculating $N = 2$ heterotic string one-loop threshold corrections with a Wilson line turned on is extended to a similar computation where $K3$ is replaced by a general Calabi–Yau 3- or 4-fold. In all cases there seems to be a generalized Kac–Moody algebra involved, whose denominator formula appears in the result. © 1999 Elsevier Science B.V. All rights reserved.

Subj. Class.: Strings
1991 MSC: 17B67; 81T30; 81R10
Keywords: $N = 2$ heterotic strings; Calabi–Yau 3- and 4-folds; Generalized Kac–Moody algebras

1. Introduction

In this paper, I extend the work of Kawai [4], calculating $N = 2$ heterotic string one-loop threshold corrections with a Wilson line turned on, to Calabi–Yau 3- and 4-folds. (See also [10] for an alternative interpretation of Kawai's result.) In full generality, this calculation provides a map from a certain class of Jacobi functions (including elliptic genera) to modular functions of certain subgroups of $\mathrm{Sp}_4(\mathbf{Q})$, in a product form. In a number of cases, these products turn out to be equal to the denominator formula of a generalized Kac–Moody algebra. It seems natural to assume that this algebra is present in the corresponding string theory, and indeed in [9] it is argued that this algebra is formed by the vertex operators of vector multiplets and hypermultiplets.

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PII S0393-0440(98)00015-1

2. Elliptic genus

In this section, I recall some basic facts about elliptic genera for Calabi–Yau manifolds, mostly from [5], and I explicitly derive it for 4-folds. Let C be a complex manifold of complex dimension d, with SU(d) holonomy. Then its elliptic genus is a function $\phi(\tau, z)$ with the following transformation properties

$$\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = e\left[\frac{mcz^2}{c\tau+d}\right]\phi(\tau, z), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}), \tag{1}$$

$$\phi(\tau, z+\lambda\tau+\mu) = (-1)^{2m(\lambda+\mu)} e[-m(\lambda^2\tau + 2\lambda z)]\phi(\tau, z), \qquad \lambda, \mu \in \mathbf{Z}, \tag{2}$$

where $m = \frac{1}{2}d$, and it has an expansion of the form

$$\phi(\tau, z) = \sum_{n\geq 0, r\in \mathbf{Z}+m} c(n, r) q^n y^r, \tag{3}$$

I use here the notations $e[x] = \mathrm{e}^{2\pi \mathrm{i} x}$, $q = e[\tau]$, $y = e[z]$. The coefficients $c(0, -m+p)$ for $0 \leq p \leq d$ have the following geometrical meaning:

$$c(0, -m+p) = \chi_p = \sum_{q=0}^{d} (-1)^{p+q} h^{p,q}, \tag{4}$$

where $h^{p,q}$ are the Hodge-numbers of C. Furthermore

$$\phi(\tau, 0) = \chi \tag{5}$$

is the Euler number of C. An important feature is that the elliptic genus can be decomposed as

$$\phi(\tau, z) = \sum_{\mu=-m+1}^{m} h_\mu(\tau)\theta_{m,\mu}(\tau, z) \tag{6}$$

for functions h_μ and $\theta_{m,\mu}$ defined by

$$h_\mu(\tau) = \sum_{N\equiv -\mu^2 (\mathrm{mod} 4m)} c_\mu(N) q^{N/4m}, \tag{7}$$

$$\theta_{m,\mu}(\tau, z) = \sum_{r\equiv \mu (\mathrm{mod} 2m)} (-1)^{r-\mu} q^{r^2/4m} y^r. \tag{8}$$

Note that the $c_\mu(N)$ are only defined for $-m+1 \leq \mu \leq m$, but since $\theta_{m,\mu+2m} = (-1)^{2m}\theta_{m,\mu}$, it is useful to define

$$c_r(N) = (-)^{r-\mu} c_\mu(N) \tag{9}$$

for all $r \equiv \mu$ mod $2m$. The relation between the coefficients of h_μ and ϕ is then given by

$$c(n, r) = c_r(4mn - r^2). \tag{10}$$

Finally, the transformation properties of the h_μ can be derived to be

$$h_\mu(\tau+1) = e\left[-\frac{\mu^2}{4m}\right] h_\mu(\tau), \tag{11}$$

$$h_\mu(-1/\tau) = \sqrt{\mathrm{i}/2m\tau} \sum_{\nu=-m+1}^{m} e\left[\frac{\mu\nu}{2m}\right] h_\nu(\tau). \tag{12}$$

Now if m is integer, the elliptic genus satisfies the defining properties of what is called a weak Jacobi form of index m and weight 0. The ring $J_{2*,*}$ of weak Jacobi forms of even weight and all indices is well known [1]. It is a polynomial algebra over M_* (the ring of ordinary modular forms) with two generators

$$A = \frac{\phi_{10,1}(\tau, z)}{\Delta(\tau)}, \qquad B = \frac{\phi_{12,1}(\tau, z)}{\Delta(\tau)}. \tag{13}$$

Here $\Delta(\tau) = \eta^{24}(\tau)$ and $\phi_{10,1}$ and $\phi_{12,1}$ are unique cusp forms of index 1 and weights 10 and 12 respectively. The generators have an expansion

$$A = y^{-1} - 2 + y + O(q), \tag{14}$$

$$B = y^{-1} + 10 + y + O(q). \tag{15}$$

It immediately follows that the space $J_{0,1}$ is one dimensional with basis B, which implies that the elliptic genus of a Calabi–Yau 2-fold is

$$\frac{\chi}{12} B. \tag{16}$$

So for $K3$, with $\chi = 24$, it should be $2B$, which is indeed the case [7]. The space $J_{0,2}$ is two dimensional, with basis $E_4(\tau)A^2$ and B^2, $E_4(\tau)$ being the normalized Eisenstein series of weight 4. So the elliptic genus is fixed by specifying χ_0 and χ, leading to

$$\chi_0 E_4 A^2 + \frac{\chi}{144}(B^2 - E_4 A^2). \tag{17}$$

In the case that the manifold has strict SU(d) holonomy, which implies that $\chi_0 = 2$ the following predictions can be done

$$\chi_1 = \frac{\chi}{6} - 8, \tag{18}$$

$$\chi_2 = \frac{2\chi}{3} + 12 \tag{19}$$

so that χ should be a multiple of 6, and there is a non-trivial relation on the Hodge-numbers

$$4(h^{1,1} + h^{3,1}) + 44 = 2h^{2,1} + h^{2,2} \tag{20}$$

as was recently noticed by Sethi et al. [6]. For a Calabi–Yau 3-fold, the elliptic genus is known to be [5]

$$\frac{\chi}{2}(y^{-1/2} + y^{1/2}) \prod_{n=1}^{\infty} \frac{(1-q^n y^2)(1-q^n y^{-2})}{(1-q^n y)(1-q^n y^{-1})}. \tag{21}$$

3. Product formulae

In this section I look at the following generalization of the formulae in [4]:

$$Z = \sum_{\mu=-m+1}^{m} Z_{m,\mu}(T, U, V, \tau) h_\mu(\tau), \tag{22}$$

where the h_μ come from a function ϕ, satisfying the transformation properties (1) and (2), and can be split like (6). For generality, I allow this function to have a pole of finite order N for $\tau \to i\infty$, but nowhere else in the fundamental domain. So the function ϕ has a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{n \geq -N, r \in \mathbf{Z}+m} c(n, r) q^n y^r, \tag{23}$$

converging for all τ with $\tau_2 > 0$ ($\tau_2 = \Im\tau$). The functions $Z_{m,\mu}$ are defined by

$$Z_{m,\mu}(T, U, V, \tau) = \sum_{m_1, m_2, n_1, n_2} \sum_{b \in 2m\mathbf{Z}+\mu} (-1)^{b-\mu} q^{(1/2)p_L^2} \bar{q}^{(1/2)p_R^2}, \tag{24}$$

$$\frac{1}{2} p_R^2 = \frac{1}{4Y} |m_1 U + m_2 + n_1 T + n_2(TU - mV^2) + bV|^2, \tag{25}$$

$$\frac{1}{2}(p_L^2 - p_R^2) = \frac{b^2}{4m} - m_1 n_1 + m_2 n_2, \tag{26}$$

$$Y = T_2 U_2 - m V_2^2. \tag{27}$$

The function Z is manifestly invariant under the following transformations

$$U \to U + 2\lambda m V + \lambda^2 m T, \qquad V \to V + \lambda T + \mu, \tag{28}$$

with $\lambda, \mu \in \mathbf{Z}$ if $m \in \mathbf{Z}$, and $\lambda, \mu \in 2\mathbf{Z}$ if $m \in \mathbf{Z} + \frac{1}{2}$. (This has the same effect as the substitutions

$$\begin{aligned} m_2 &\to m_2 - \mu^2 m n_2 + \mu b, \\ n_1 &\to n_1 + \lambda^2 m m_1 - 2\lambda\mu m n_2 + \lambda b, \\ b &\to b + 2\lambda m m_1 - 2\mu m n_2, \end{aligned} \tag{29}$$

and these leave the inproduct $b^2/4m - m_1 n_1 + m_2 n_2$ invariant. In the same way one proves the other invariances.) It is also invariant under the generalization of $\mathrm{SL}(2, \mathbf{Z})_T \times \mathrm{SL}(2, \mathbf{Z})_U$, generated by

$$T \to T + 1, \tag{30}$$

$$T \to -\frac{1}{T}, \quad U \to U - m\frac{V^2}{T}, \quad V \to \frac{V}{T}, \tag{31}$$

$$U \to U + 1, \tag{32}$$

$$U \to -\frac{1}{U}, \quad T \to T - m\frac{V^2}{U}, \quad V \to \frac{V}{U}. \tag{33}$$

Furthermore, it is invariant under exchange of T and U, and under a parity transformation

$$T \leftrightarrow U, \tag{34}$$

$$V \to -V. \tag{35}$$

These transformations generate a group isomorphic to $\mathrm{Sp}_4(\mathbf{Z})$ if $m = 1$, and to a paramodular subgroup of $\mathrm{Sp}_4(\mathbf{Q})$ [3] for $m > 1$. Since $\tau_2 Z$ is invariant under modular transformation of τ, as will be shown later, the following integral is well-defined and can be evaluated explicitly by the methods of [8,9]:

$$\mathcal{I} = \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2}(Z - c(0,0)). \tag{36}$$

The subtraction is to remove the logarithmic singularities due to the massless hypermultiplets, and is needed only if m is integer. If it is not, I define $c(0,0)$ to be zero. Poisson resummation on m_1, m_2 leads to

$$\sum_{m_1,m_2} q^{(1/2)p_L^2}\bar{q}^{(1/2)p_R^2} = \sum_{k_1,k_2} \frac{Y}{U_2\tau_2} q^{(1/4m)b^2} \exp\mathcal{G}, \tag{37}$$

where

$$\mathcal{G} = \frac{-\pi Y}{U_2^2\tau_2}|A|^2 - 2\pi \mathrm{i} T(n_1k_2 + n_2k_1) + \frac{\pi b}{U_2}(V\tilde{A} - \bar{V}A)$$
$$- \frac{\pi m n_2}{U_2}(V^2\tilde{A} - \bar{V}^2 A) + \frac{2\pi \mathrm{i} m V_2^2}{U_2^2}(n_1 + n_2\bar{U})A, \tag{38}$$

$$A = -k_1 + n_1\tau + k_2 U + n_2\tau U, \tag{39}$$

$$\tilde{A} = -k_1 + n_1\tau + k_2\bar{U} + n_2\tau\bar{U}. \tag{40}$$

By applying another Poisson resummation on b, it is easy to find the following transformation properties of $Z_{\mu,m}$:

$$Z_{\mu,m}(-1/\tau) = \sqrt{\tau/2m\mathrm{i}} \sum_{\nu=-m}^{m} e\left[\frac{-\mu\nu}{2m}\right] Z_{\nu,m}(\tau), \tag{41}$$

which together with the known properties (12) of the h_μ prove the modular invariance of $\tau_2 Z$. Following [8,9] a bit further I find

$$\mathcal{I}_0 = \frac{Y}{U_2}\int \frac{\mathrm{d}^2\tau}{\tau_2^2}\phi(\tau,0) = \frac{\pi Y}{3U_2}E_2(\tau)\phi(\tau,0)|_{q^0}, \tag{42}$$

$$\mathcal{I}_d = \sum_{b\in\mathbf{Z}+m} 2\pi c(0,b)\left[b^2\frac{V_2^2}{U_2} - |b|V_2 + \frac{U_2}{6}\right] - c(0,0)\ln(kY)$$
$$- \ln \prod_{(l>0,b\in\mathbf{Z}+m),(l=0,0<b\in\mathbf{Z}+m)} |1 - e[lU + bV]|^{4c(0,b)}. \tag{43}$$

(This under the assumption that $0 \leq V_2 \leq U_2/|b|$ for all b with $c(0, b) \neq 0$). Here

$$k = \frac{8\pi}{3\sqrt{3}} \mathrm{e}^{1-\gamma}, \tag{44}$$

$$\mathcal{I}_{nd} = -\ln \prod_{k>0, l\in\mathbf{Z}, b\in\mathbf{Z}+m} |1 - e[kT + lU + bV]|^{4c(kl,b)} \tag{45}$$

(This for T_2 large enough). Putting this all together, I obtain

$$\mathcal{I} = -2\ln(kY)^{(1/2)c(0,0)} \times \left| e[pT + qU + rV] \prod_{(k,l,b)>0} (1 - e[kT + lU + bV])^{c(kl,b)} \right|^2, \tag{46}$$

where the coefficients p, q, r are given by

$$p = \sum_{b\in\mathbf{Z}+m} \frac{b^2}{4m} c(0, b), \tag{47}$$

$$q = \sum_{b\in\mathbf{Z}+m} \frac{1}{24} c(0, b), \tag{48}$$

$$r = \sum_{b\in\mathbf{Z}+m} -\frac{|b|}{4} c(0, b), \tag{49}$$

and the summation condition means $k > 0$ or $k = 0, l > 0$ or $k = l = 0, b > 0$ (always with $k, l \in \mathbf{Z}$ and $b \in \mathbf{Z} + m$). In the calculation I use the following non-trivial identity:

$$\sum_b \frac{b^2}{4m} c(0, b) = \frac{E_2(\tau)\phi(\tau, 0)|_{q^0}}{24}. \tag{50}$$

This can be proven as follows (cf. [11]). First note that

$$\sum_\mu h_\mu(\tau)\theta'_{m,\mu}(\tau, z) = \frac{1}{4m} \sum r^2 c(n, r) q^n y^r \tag{51}$$

(where $\theta' = q\frac{\partial}{\partial q}\theta$). So it is equivalent to prove that the constant term of the following expression vanishes:

$$\sum_\mu h_\mu(\tau)\theta'_{m,\mu}(\tau, 0) - \frac{E_2(\tau)h_\mu(\tau)\theta_{m,\mu}(\tau, 0)}{24}. \tag{52}$$

But this can be rewritten as

$$\sum_\mu (h_\mu(\tau)\eta(\tau))(\theta_{m,\mu}(\tau, 0)\eta^{-1}(\tau))'. \tag{53}$$

This function transforms as a modular function of weight 2, so multiplying it by $\mathrm{d}\tau$ gives an $\mathrm{SL}_2(\mathbf{Z})$ invariant differential form. By assumption, it can have a pole at $\tau \to \mathrm{i}\infty$, but nowhere else in the fundamental domain. A contour integral argument then shows that the residue of this pole must vanish. But this is just the constant term of the function above.

Applying these formulae to $2B$, the elliptic genus of $K3$, I recover the result of Kawai [4]. Now consider the elliptic genus of a Calabi–Yau 4-fold,

$$\phi = \chi_0 E_4 A^2 + \frac{\chi}{144}(B^2 - E_4 A^2). \tag{54}$$

Amazingly, the χ-dependent part equals the coefficients of Gritsenko and Nikulin's second product formula [3], which is known to be associated to the generalized Kac–Moody algebra which is an automorphic form correction to the Kac–Moody algebra defined by the symmetrized generalized Cartan matrix

$$G_2 = \begin{pmatrix} 4 & -4 & -12 & -4 \\ -4 & 4 & -4 & -12 \\ -12 & -4 & 4 & -4 \\ -4 & -12 & -4 & 4 \end{pmatrix}. \tag{55}$$

Unfortunately, there is no such formula for the χ_0-dependent part. So for a Calabi–Yau 4-fold I find

$$\mathcal{I} = -\chi_0 \ln((kY)^6 |\Pi_6(\Omega)|^2) - \tfrac{1}{3}\chi \ln((kY)^2 |F_2(\Omega)|^2), \tag{56}$$

where F_2 is Gritsenko and Nikulins product and Π_6 is

$$e[2V] \prod_{(k,l,b)>0} (1 - e[kT + lU + bV])^{c(kl,b)} \tag{57}$$

of weight 6, with coefficients c coming from $2E_4A^2$. The following section describes the product formula for a Calabi–Yau 3-fold.

4. Calabi–Yau 3-folds

In this section I apply my formulae to Eq. (21), without the factor $\frac{1}{2}\chi$. Expanding this in q gives

$$(y^{-1/2} + y^{1/2}) + O(q) \tag{58}$$

so that $c(0, -\frac{1}{2}) = c(0, \frac{1}{2}) = 1$, and the corresponding product formula reads

$$F_0(T, U, V) = p^{1/12} q^{1/12} y^{-1/4} \prod_{(k,l,b)>0} (1 - p^k q^l y^b)^{c(kl,b)} \tag{59}$$

of weight 0, where now $p = e[T]$, $q = e[U]$, $y = e[V]$. In the limit $V \to 0$, this product behaves like

$$V\eta^2(p)\eta^2(q) \tag{60}$$

as can be expected for $\chi = 2$. This product can be expanded in terms of p (since it is valid for T_2 large enough). It turns out to be useful to consider $F_0(T, U, 2V)$. Thus

$$F_0(T, U, 2V) = \sum_{k \in \mathbf{Z}_{\geq 0} + 1/12} \phi_k(q, y) p^k \tag{61}$$

This is a variant of what is known as a Fourier–Jacobi expansion. The transformation properties of $F_0(T, U, V)$ imply that the coefficients ϕ_m should be Jacobi forms of weight 0 and index $6k$, with a possible multiplier system. From the product formula it is possible to read of the lowest order coefficient

$$\begin{aligned}\phi_{1/12}(q, y) &= q^{1/12}(y^{-1/2} - y^{1/2}) \prod_{n>0}(1 - q^n y)(1 - q^n y^{-1}) \\ &= \theta_{11}(q, y)\eta^{-1}(q)\end{aligned} \tag{62}$$

by the product formula for theta-functions. This is indeed a Jacobi cusp form of weight 0 and index $\frac{1}{2}$ with multiplier system [2], which can serve as a consistency check. It can be written as a sum as follows

$$\left(\sum_{n\in\mathbf{Z}}(-1)^n q^{(2n+1)^2/8} y^{(2n+1)/2}\right)\left(\sum_{n\geq 0} p(n) q^{n-1/24}\right), \tag{63}$$

where $p(n)$ is the partition function. Now unlike the case of $F_2(\Omega)$ from [3], it does not seem to be possible to write the entire product as a lifting of its first Fourier–Jacobi coefficient. It does seem to be likely that this function is also related to some generalized Kac–Moody algebra. This is under investigation. The final result for the Calabi–Yau 3-fold calculation is

$$\mathcal{I} = -\chi \ln |F_0(\Omega)|^2. \tag{64}$$

Acknowledgements

I would like to thank R. Dijkgraaf for helpful discussions.

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